Fundamentals of Stochastic Processes 048868

Home Assignment 2

Weak convergence. Conditional expectation. Regular conditional distributions.

Submit until: April 6

1. Weak convergence:

   (a) Let $X_1, X_2, \ldots,$ be i.i.d. $N(0,1)$ (standard normal). Let $M_n = \max_{m \le n} X_m$.

   i. By using the following inequalities, that hold for all $x > 0$:

   $$(x^{-1} - x^{-3})e^{-x^2/2} \le \int_x^\infty e^{-y^2/2} dt \le x^{-1} e^{-x^2/2},$$

   show that for any θ,

   $$\lim_{x\to\infty} \frac{P\left(X_i > x + \frac{\theta}{x}\right)}{P(X_i > x)} = e^{-\theta}.$$

   ii. For $n = 1, 2, \ldots,$ let b_n be the unique number with $P(X_i > b_n) = 1/n$. Show

   $$\lim_{n\to\infty} P(b_n(M_n - b_n) \le x) = \exp(-e^{-x}).$$

   iii. Show that $b_n \sim (2\log n)^{1/2}$ and conclude $M_n/(2\log n)^{1/2} \to 1$ in probability.

   (b) Let $X_n, 1 \le n \le \infty$ be integer valued. Show that $P_{X_n} \Rightarrow P_{X_\infty}$ if and only if $P(X_n = m) \to_{n\to\infty} P(X_\infty = m)$ for all m.

   (c) Show that if $X_n = (X_n^1, \ldots, X_n^n)$ is uniformly distributed over the surface of the sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$ then $P_{X_n^1} \Rightarrow N(0,1)$. Hint: Let $Y_1, Y_2, \ldots$ be i.i.d. $N(0,1)$ and define $X_n^i = Y_i(n/\sum_{m=1}^n Y_m^2)^{1/2}$.

2. Conditional expectation:

   (a) Let $\mathrm{Var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

   $$\mathrm{Var}(X) = E(\mathrm{Var}(X|\mathcal{F})) + \mathrm{Var}(E(X|\mathcal{F})).$$

   (b) Show that if $E(Y|\mathcal{G}) = X$ and $EX^2 = EY^2 < \infty$ then $X = Y$ a.s.

   (c) Give an example on $\Omega = \{a, b, c\}$ in which

   $$E(E(X|\mathcal{F}_1)|\mathcal{F}_2) \ne E(E(X|\mathcal{F}_2)|\mathcal{F}_1).$$