Fix a complete metric space with metric $d$.

**Definition 0.1** $T$ is a contraction operator with constant $K < 1$ if for any $x, y$

$$d(Tx, Ty) \leq Kd(x, y).$$  \hspace{1cm} (0.1)

Let $T$ be a contraction operator.

**Claim 0.2** $T$ is continuous.

**Proof.** Let $x_n \to x$, that is $d(x_n, x) \to 0$. Then

$$d(Tx_n, Tx) \leq Kd(x_n, x) \to 0$$  \hspace{1cm} (0.2)

and the result is established. \hfill \blacksquare

**Claim 0.3** For any $x$, the sequence of iterates $T^nx$ converges to a fixed point.

**Proof.** First we show that $T^nx$ is a Cauchy sequence. Assume $n > m$. We have

$$d(T^nx, T^mx) \leq K^m d(T^{n-m}x, x)$$  \hspace{1cm} (0.3)

$$\leq K^m \left[ d(T^{n-m}x, T^{n-m-1}x) + d(T^{n-m-1}x, x) \right]$$  \hspace{1cm} (0.4)

by the triangle inequality

$$\leq K^m \left[ d(T^{n-m}x, T^{n-m-1}x) + \ldots + d(Tx, x) \right]$$  \hspace{1cm} (0.5)

$$\leq K^m \left[ K^{n-m-1} d(Tx, x) + \ldots + d(Tx, x) \right]$$  \hspace{1cm} (0.6)
now bound the sum of powers of $K$ by an infinite geometric sum

$$\leq K^{m} \frac{1}{1-K} d(Tx,x) \to 0 \quad (0.7)$$

as $m \to \infty$. Thus by completeness $T^m x \to x_0$ for some limit $x_0$. That is, $d(T^m x, x_0) \to 0$. Now since $T$ is continuous, this implies that also $d(TT^m x, Tx_0) = d(T^{m+1} x, Tx_0) \to 0$. By the triangle inequality,

$$d(Tx_0, x_0) \leq d(Tx_0, T^{n+1} x_0) + d(T^{n+1} x_0, x_0) \quad (0.8)$$

However as we just argued, the first term on the right $\to 0$ by continuity, while the second $\to 0$ since as we proved $T^m x \to x_0$. Thus $d(Tx_0, x_0) = 0$ so that $x_0$ is a fixed point of $T$. 

**Claim 0.4** The fixed point is unique, and thus independent of the starting point of the iterates.

**Proof.** Suppose there are two fixed points $x_0, x_1$. Then

$$Kd(x_0, x_1) \geq d(Tx_0, Tx_1) \quad (0.9)$$

since $T$ is a contraction. However, $Tx_0 = x_0, Tx_1 = x_1$ since they are fixed points. So

$$= d(x_0, x_1) \quad (0.10)$$

which, since $K < 1$, is possible only if $d(x_0, x_1) = 0$, that is $x_0 = x_1$. 

**Claim 0.5** The iterates converge geometrically, that is,

$$d(T^m x, x_0) \leq K^m d(x, x_0). \quad (0.11)$$

This means that the rate of convergence is at least as fast as $K$, with a constant that depends on the initial point.
**Proof.** Since $x_0$ is a fixed point, $x_0 = T^n x_0$ so

\begin{align*}
d(T^n x, x_0) &= d(T^n x, T^n x_0) \quad \text{(0.12)} \\
&\leq K^n d(x, x_0). \quad \text{(0.13)}
\end{align*}

Exercise 0.6  *Suppose $T$ is not a contraction, but is continuous and there exists $K < 1$ and $J > 0$ so that $T^J$ is a contraction with constant $K^J$. Show that all the results above continue to hold.*