Home Assignment 5

Submit Part I by May 25, Part II by June 8.

Ito calculus

Part I: Stochastic integrals

1. For the stochastic integral, prove properties 1–6 (from the lectures) for processes in $E$.

2. For processes in $L^2$: If $Y_n \rightarrow Y$ in $L^2$, show that $E_n \rightarrow EY$. Now prove property 1.

3. Show that if $X_n \rightarrow X$ in $L^2$ and $Y_n \rightarrow Y$ in $L^2$ then $X_n + Y_n \rightarrow X + Y$ in $L^2$. Now prove property 2 in $L^2$.

4. Prove property 3 in $L^2$.

Part II: Stochastic Differential Equations

5. Let $X_t = W_t$ and $Y_t = e^{W_t^2}$. Show that $(X_t, Y_t)$ solves the set of stochastic differential equations

$$dX_t = dW_t, \quad X_0 = 0,$$

$$dY_t = 2X_tY_t dW_t + (Y_t + 2X_t^2 Y_t) dt, \quad Y_0 = 1.$$

6. Let $W^1, \ldots, W^n$ be independent BMs and denote $W = (W^1, \ldots, W^n)$ ($W$ is called an $n$-dimensional BM). Let

$$R = |W| = \left( \sum_{i=1}^{n} (W^i)^2 \right)^{1/2}.$$

Show that $R$ solves the stochastic Bessel equation:

$$dR = \sum_i \frac{W^i}{R} dW^i + \frac{n-1}{2R} dt.$$

7. Let $W$ be an $n$-dimensional BM, for $n \geq 3$. Write $X = W + x_0$ where the point $x_0$ lies in the region $U = \{0 < R_1 < |x| < R_2\}$. Calculate explicitly the probability that $X$ will hit the outer sphere $\{|x| = R_2\}$ before hitting the inner sphere $\{|x| = R_1\}$. Hint: Check that $\Phi(x) = |x|^{2-n}$ satisfies $\Delta \Phi = 0$ for $x \neq 0$. Modify $\Phi$ to build a function $u$ which equals 0 on the inner sphere and 1 on the outer sphere.